

MTH 203: Introduction to Groups and Symmetry

Homework III

(Due 5/9/2022)

Problems for submission

1. Establish the assertion in 1.4(iv) of the Lesson Plan.
2. Establish the assertion in 2.2(i) of the Lesson Plan.
3. Use the fact that \mathbb{R}^n is a group with respect to component-wise addition to show that every proper subgroup H that is also a subspace of \mathbb{R}^2 is a line through the origin. Also, describe a typical coset of such a subgroup H .
4. Show that if a group has no proper subgroups, it must be cyclic of prime order.
5. Show that if in a group G , every non-identity element is of order 2, then G is abelian.

Problems for practice

1. Show that if a and b are integers with greatest $\gcd(a, b) = d$, then there exist integers x and y such that $ax + by = d$. (This is known as the Bézout's identity.)
2. Show that the following properties hold in a group G .

(a) For $g \in G$, we have $(g^{-1})^{-1} = g$.

(b) For $g' \in G$, we have

$$g'G := \{g'g : g \in G\} = G.$$

(c) For $g \in G$ and $a, b \in \mathbb{Z}$, we have

$$g^a g^b = g^{a+b}.$$

(d) For $g \in G$ and $a, b \in \mathbb{Z}$, we have

$$g^{ab} = (g^a)^b = (g^b)^a.$$

(e) For $g_1, g_2, \dots, g_k \in G$, we have

$$(g_1 g_2 \dots g_k)^{-1} = g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}.$$

3. Show that the following sets form groups.
 - (a) $O(n, F) = \{A \in \text{GL}(n, F) : AA^T = A^T A = I_n\}$. (This is called the *orthogonal group in dimension n* .)
 - (b) $SO(n, F) = \{A \in O(n, F) : \det(A) = 1\}$. (This is called the *special orthogonal group in dimension n* .)
4. Let G be an abelian group.
 - (a) Show that $H = \{a \in G : a^2 = 1\}$ is a subgroup of G . Give an example of a nonabelian group in which H is not a subgroup.
 - (b) Show that $H_n = \{a^n : a \in G\}$ is a subgroup of G .